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On the Susceptibility and Clustering Properties of Unbounded Spins

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We prove that unbounded spin systems with superstable two-body interactions have generalized susceptibilities which are strictly positive. This result is then used to prove that the decay of the correlations cannot be faster than the decay of the potential if the potential decays with a power law.

KEY WORDS: Unbounded spins; correlations; clustering; susceptibility.

1. INTRODUCTION

Unbounded spin systems with superstable interactions have very nice properties which have been used to established the existence and the uniqueness of the equilibrium states[1,2] and to discuss the decay properties of the correlation functions.[3–6]

In this note, we use these same properties to show that the generalized susceptibilities χ_d are strictly positive (Section 4), where

$$\chi_d = \sum_{y \in \mathbb{Z}^\nu} (\langle d_x d_y \rangle - \langle d_x \rangle \langle d_y \rangle)$$

with $d(q)$ an arbitrary function of the spin variable which satisfies the condition that $|d(q)| \leqslant c\,|q|$. In Section 5, we then consider two-body potentials which are integrable and which have a power-law decay at

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infinity; under reasonable conditions, it is shown that if the correlation functions are assumed to decay faster than the potential then $\chi_d = 0$. It thus follows from the previous result that the decay of the correlation functions cannot be faster than the decay of the potential. In Section 3 it is shown that for finite volume there exist certain canonical states which obey canonical sum rules of which $\chi_d = 0$ is a special case. These sum rules which are thus not valid for infinite systems if the potential is integrable are similar to those obtained for continuous Coulomb systems[7]; it is expected that they will hold whenever the potential decays slower than or as $|x|^{-(\nu-1)}$, as $|x| \to \infty$, with ν the dimension of the lattice. The superstability estimates necessary for the proofs are given in Section 2.

2. INFINITE SYSTEMS EQUILIBRIUM STATES: DEFINITIONS

We consider a classical lattice system defined on $\mathbb{Z}^\nu$. At each site $x \in \mathbb{Z}^\nu$ is associated a random spin variable q_x, which takes values in $\mathbb{R}^d$, $d \in \mathbb{N}$. The configuration space of the system is thus

$$\Omega = \{\mathbf{q} = (q_x)_{x \in \mathbb{Z}^\nu}\}$$

and q_Λ, for $\Lambda \subset \mathbb{Z}^\nu$, is the restriction of $\mathbf{q}$ to the region Λ. Ω is a topological space with product topology inherited from $\mathbb{R}^d$. We denote by $M(\Omega)$ the set of Borel probability measures on Ω, with the topology determined by the continuous bounded and cylindrical functions on Ω.

For each spin q_x a free measure $\lambda(dq_x)$ is given where $\lambda(dq)$ is a positive Borel probability measure on $\mathbb{R}^d$. The one-body potential is included in the free measure which is thus dependent on β, the inverse temperature. We suppose that there exists $r > 0$ such that

$$\int \lambda(dq)\, e^{r\beta q^2} < \infty \tag{2.1}$$

We also suppose that $\lambda(dq)$ is not concentrated on a single value of q, that is

$$\lambda(dq) \neq \delta(q - q_0) \qquad \forall q_0 \in \mathbb{R}^d \tag{2.2}$$

The interaction is defined by means of a two-body potential $\phi_{xy}(q_x, q_y)$, where $\phi_{xy}(q, q')$ is a real function on $\mathbb{R}^d \times \mathbb{R}^d$ which is assumed to satisfy the condition

$$|\phi_{xy}(q, q')| \leqslant J(|x - y|)\, |q|\, |q'| \tag{2.3}$$

where $J(n)$ is a decreasing function of $n \in \mathbb{N}$, such that

$$J(n) \leqslant K n^{-\nu-\varepsilon} \tag{2.4}$$

for some K and ε positive.

Equations (2.1) and (2.3) easily imply that the system belongs to the class of systems for which the superstability estimates are valid[1] if we add the assumption

$$r > J/2 \tag{2.5}$$

where

$$J = \sum_{0 \neq x \in \mathbb{Z}^\nu} J(|x|) \tag{2.6}$$

These systems have some nice properties, exploited in Refs. 1 and 2. The main one is the following: Let $M_0(\Omega) \subset M(\Omega)$ be the set of probability measures μ with the following properties:

(1) μ is a DLR measure, i.e., for each finite $\Lambda \subset \mathbb{Z}^\nu$ the conditional probability $\mu[dq_\Lambda \mid q_{\Lambda^c}]$ satisfies the equilibrium equations

$$\mu[dq_\Lambda \mid q_{\Lambda^c}] = Z_\Lambda(q_{\Lambda^c})^{-1} \lambda(dq_\Lambda) \exp[-\beta H(q_\Lambda \mid q_{\Lambda^c})] \tag{2.7}$$

μ a.e. with respect to the conditioning spins q_{Λ^c}. Here $Z_\Lambda(q_{\Lambda^c})$ is the normalization constant, $\lambda(dq_\Lambda) = \prod_{x \in \Lambda} \lambda(dq_x)$ and

$$H(q_\Lambda \mid q_{\Lambda^c}) = \frac{1}{2} \sum_{\substack{x, y \in \Lambda \\ x \neq y}} \phi_{xy}(q_x, q_y) + \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} \phi_{xy}(q_x, q_y) \tag{2.8}$$

(2) If we define

$$R_N = \left\{ \mathbf{q} \in \Omega \;\middle|\; \sum_{|x| \leqslant j} q_x^2 \leqslant N^2 (2j+1)^\nu, \ \forall j \geqslant 0 \right\} \tag{2.9}$$

then

$$\mu\left(\bigcup_{N>0} R_N \right) = 1 \tag{2.10}$$

Notice that Eqs. (2.7) imply that $H_\Lambda(q_\Lambda \mid q_{\Lambda^c})$ is (almost everywhere) well defined. In particular the function

$$W(q_x; q_{\mathbb{Z}^\nu/x}) = H(q_x \mid q_{\mathbb{Z}^\nu/x}) \tag{2.11}$$

is well defined.

In Refs. 1 and 2, the following theorem is proved.

Theorem 2.1. If conditions (2.1), (2.3), (2.5) are satisfied, the set $M_0(\Omega)$ is not empty. Furthermore, for any $\gamma > 0$ such that

$$\gamma < \beta(r - J/2) \tag{2.12}$$

there exists a constant δ such that for any finite $\Lambda \subset \mathbb{Z}^\nu$ and any $\mu \in M_0(\Omega)$

$$\mu(dq_\Lambda) \leqslant \left[\prod_{x \in \Lambda} \lambda(dq_x)\, e^{r\beta q_x^2} \right] \exp\left[-\sum_{x \in \Lambda} (\gamma q_x^2 - \delta) \right] \tag{2.13}$$

Remarks

(1) Theorem 2.1 is valid under a condition weaker than (2.5). In fact, it is sufficient that there exist two positive constants A and B such that for any finite $\Lambda \subset \mathbb{Z}^\nu$,

$$\sum_{x \in \Lambda} r q_x^2 + \frac{1}{2} \sum_{\substack{x, y \in \Lambda \\ x \neq y}} \phi_{xy}(q_x, q_y) \geqslant \sum_{x \in \Lambda} (A q_x^2 - B) \tag{2.14}$$

However, in the following we shall need a bound on the interaction also stronger than Eq. (2.5) [see Eq. (4.7)].

(2) Equation (2.3) implies that $\phi_{xy}(q, q') = 0$ if q or q' is zero; this condition can always be realized by changing the free measure. If the potential satisfies the condition

$$|\phi_{xy}(q, q')| \leqslant J(|x - y|)\, |q|^\alpha\, |q'|^\alpha$$

a change of variable will reduce this model to the case treated in the paper.

3. SUM RULES FOR FINITE SYSTEMS

Let Λ be a finite subset of $\mathbb{Z}^\nu$ and let $\Omega_\Lambda = \{q_\Lambda = (q_x)_{x \in \Lambda}\}$ be the configuration space of a "finite system in the volume Λ." For any function $d(q)$ on $\mathbb{R}^d$ such that $|d(q)| \leqslant c\,|q|$, we define a "canonical state D_Λ" for the system in the finite volume Λ be a probability measure μ_Λ on Ω_Λ such that

$$\mu_\Lambda(dq_\Lambda) = \left[\prod_{x \in \Lambda} \lambda(dq_x) \right] \delta\left(D_\Lambda - \sum_{x \in \Lambda} d(q_x) \right) \rho_\Lambda(q_\Lambda) \tag{3.1}$$

where $\rho_\Lambda(q_\Lambda)$ is a measurable function on Ω_Λ, integrable with respect to $\lambda(dq_\Lambda)$.

Introducing the correlation functions

$$\rho_X^{(\Lambda)}(q_X) = \int_{\Omega_{\Lambda \backslash X}} \prod_{x \in \Lambda \backslash X} \lambda(dq_x)\, \delta\left(D_\Lambda - \sum_{x \in \Lambda} d(q_x) \right) \rho_\Lambda(q_\Lambda) \tag{3.2}$$

where $X \subset \Lambda$, we have

$$\left\langle \prod_{i=1}^{k} \prod_{j=1}^{d} (q_{x_i})_j^{n_{ij}} \right\rangle^{(\Lambda)} = \int \prod_{x \in X} \lambda(dq_x)\, \rho_X^{(\Lambda)}(q_X) \prod_{i=1}^{k} \prod_{d=1}^{d} (q_{x_i})_j^{n_{ij}} \tag{3.3}$$

where $X = (x_1, ..., x_k)$, $n_{ij} \in \mathbb{N}$, $\langle \cdots \rangle^{(\Lambda)}$ denotes the expectation with respect to μ_Λ and we are supposing that all the moments of μ_Λ are finite. It follows immediately from these definitions that μ_Λ satisfies the following "sum rules," analogous to those derived in Ref. 7 for continuous systems:

$$\sum_{y \notin X} \int \lambda(dq_y)\, d(q_y) [\rho_{Xy}^{(\Lambda)}(q_X q_y) - \rho_X^{(\Lambda)}(q_X)\, \rho_y^{(\Lambda)}(q_y)[$$
$$= - \sum_{x \in X} [d(q_x) - \langle d(q_x)\rangle^{(\Lambda)}]\, \rho_X^{(\Lambda)}(q_X) \tag{3.4}$$

which imply in particular the following identity:

$$\sum_{y \in \Lambda} [\langle d(q_x)\, d(q_y)\rangle^{(\Lambda)} - \langle d(q_x)\rangle^{(\Lambda)} \langle d(q_y)\rangle^{(\Lambda)}] = 0 \tag{3.5}$$

The equilibrium states of the infinite system are defined by correlation functions $\rho_X(q_X)$ which are solutions of some equilibrium equations, such as DLR, Kirkwood–Salzburg, or other equivalent equations. It is expected and often can be proved that the equilibrium states of the infinite system coincide with the thermodynamic limit $\Lambda \to \mathbb{Z}^\nu$ with $|\Lambda|^{-1} D_\Lambda$ fixed. We are therefore led to ask whether or not the sum rules (3.4) will still be valid for the infinite systems introduced in Section 2. We shall show in the following that these sum rules cannot be valid for systems with forces satisfying the integrability condition (2.4). This result is similar to the one obtained for continuous systems.[8]

4. SUSCEPTIBILITY OF THE INFINITE SYSTEM

Let us consider an infinite system satisfying the conditions discussed in Section 1 and let $\mu \in M_0(\Omega)$. With $\Lambda \subset \mathbb{Z}^\nu$ a finite set, we define

$$Q_\Lambda = \sum_{x \in \Lambda} q_x \tag{4.1}$$

From Theorem 2.1 it immediately follows that all the powers of Q_Λ are integrable with respect to μ. Then it has a meaning to consider the quantity

$$\chi_\Lambda = \frac{1}{|\Lambda|} [\langle Q_\Lambda^2\rangle - \langle Q_\Lambda\rangle^2] \tag{4.2}$$

where $\langle \cdots \rangle$ denotes the expectation with respect to μ.

Suppose that μ is translation invariant and that

$$\rho^T(x-y) = \langle q_x q_y \rangle - \langle q_x \rangle \langle q_y \rangle \tag{4.3}$$

satisfies the integrability condition:

$$\sum_{z \in \mathbb{Z}^\nu} \rho^T(z) < \infty \tag{4.4}$$

Then it is easy to show that χ_Λ has a limit as $\Lambda \to \mathbb{Z}^\nu$ and that

$$\chi = \lim_{\Lambda \to \mathbb{Z}^\nu} \chi_\Lambda = \sum_{z \in \mathbb{Z}^\nu} \rho^T(z) \tag{4.5}$$

χ is called the susceptibility of the system in the state μ. Then, if the sum rule (3.5) were valid for the infinite system, the susceptibility would be zero. In this section, we want to show that this is not the case for integrable potential and thus the sum rule cannot hold. In fact we shall prove the following theorem.

Theorem 4.1. Let us assume that a classical lattice system satisfies the conditions (2.1)–(2.4) and the condition [which implies (2.5)]:

$$r > \frac{J}{2} + J(1) \tag{4.6}$$

Then, if $\mu \in M_0(\Omega)$, there exists a constant $M > 0$ such that,for any finite $\Lambda \subset \mathbb{Z}^\nu$

$$\chi_\Lambda \geqslant M \tag{4.7}$$

In order to prove the theorem we need some lemmas where we use some ideas imployed in Ref. 9 by Ginibre for a similar problem.

In the following, we shall always suppose that the conditions (2.1)–(2.4) and (4.6) are satisfied, that $\mu \in M_0(\Omega)$ and that $\Lambda \subset \mathbb{Z}^\nu$ is a finite set. We add also the condition

$$\int \lambda(dq)q = 0 \tag{4.8}$$

which simplifies the notation; it is not restrictive since χ does not change if all the spins are translated by the same quantity, i.e., if the define

$$Q_\Lambda = \sum_{x \in \Lambda} \left[q_x - \int \lambda(dq) q \right]$$

Lemma 4.1. With $W(q_x; q_{\mathbb{Z}^\nu/x})$ the function defined in Eq. (2.11), then the function on Ω

$$A_\Lambda(\mathbf{q}) = \sum_{x \in \Lambda} A_x(\mathbf{q}) = \sum_{x \in \Lambda} \int \lambda(d\bar{q}_x)\bar{q}_x \exp\{-\beta[W(\bar{q}_x; q_{\mathbb{Z}^\nu/x}) - W(q_x; q_{\mathbb{Z}^\nu/x})]\} \tag{4.9}$$

is summable with respect to μ and

$$\langle Q_\Lambda \rangle = \langle A_\Lambda \rangle \tag{4.10}$$

Proof. Equations (2.7), (2.8), (4.1) and Fubini's theorem imply that

$$\begin{aligned}\langle Q_\Lambda \rangle &= \sum_{x \in \Lambda} \int \mu[dq_{\Lambda^c}] \int \lambda(dq_\Lambda)\, Z_\Lambda(q_{\Lambda^c})^{-1} \exp\{-\beta H(q_\Lambda | q_{\Lambda^c})\} q_x \int \lambda(d\bar{q}_x) \\ &= \sum_{x \in \Lambda} \int \mu[dq_{\Lambda^c}] \int \lambda(dq_\Lambda)\, Z_\Lambda(q_{\Lambda^c})^{-1} \exp\{-\beta H(q_\Lambda | q_{\Lambda^c})\} \\ &\quad \times \int \lambda(d\bar{q}_x)\bar{q}_x \exp\{-\beta[W(\bar{q}_x; q_{\mathbb{Z}^\nu/x}) - W(q_x; q_{\mathbb{Z}^\nu/x})]\}\end{aligned}$$

where we used the fact that $\int \lambda(dq) = 1$. ∎

Lemma 4.2. $Q_\Lambda A_\Lambda(\mathbf{q})$ is summable with respect to μ and

$$\langle Q_\Lambda^2 \rangle - \langle Q_\Lambda A_\Lambda \rangle = \langle B_\Lambda \rangle \tag{4.11}$$

where

$$B_\Lambda = \sum_{x \in \Lambda} q_x^2 \tag{4.12}$$

Proof. Using Eqs. (4.1), (4.8), and (4.12), we find

$$\begin{aligned}\langle Q_\Lambda^2 \rangle - \langle B_\Lambda \rangle &= \left\langle \left(\sum_{x \in \Lambda} q_x \right)^2 - \sum_{x \in \Lambda} \left[q_x^2 - q_x \int \lambda(d\bar{q}_x)\bar{q}_x \right] \right\rangle \\ &= \left\langle \sum_{\substack{x_1, x_2 \in \Lambda \\ x_1 \neq x_2}} q_{x_1} q_{x_2} + \sum_{x \in \Lambda} q_x \int \bar{q}_x \lambda(d\bar{q}_x) \right\rangle\end{aligned}$$

Proceeding as in Lemma 4.1, it is easy to show that, if $x_1 \neq x_2$

$$\langle q_{x_1} q_{x_2} \rangle = \left\langle q_{x_1} \int \lambda(d\bar{q}_{x_2})\bar{q}_{x_2} \exp\{-\beta[W(\bar{q}_{x_2}; q_{\mathbb{Z}^\nu/x_2}) - W(q_{x_2}; q_{\mathbb{Z}^\nu/x_2})]\} \right\rangle$$

and that

$$\left\langle q_{x_1} \int \bar{q}_{x_1} \lambda(d\bar{q}_{x_1}) \right\rangle = \left\langle q_{x_1} \int \lambda(d\bar{q}_{x_1}) \bar{q}_{x_1} \exp\{-\beta[W(\bar{q}_{x_1}; q_{\mathbb{Z}^\nu/x_1}) - W(q_{x_1}; q_{\mathbb{Z}^\nu/x_1})]\} \right\rangle$$

These equations immediately imply (4.11). ∎

Lemma 4.3. $A_x(\mathbf{q})^2$ is summable with respect to μ and there exists a positive constant M_1 such that

$$\langle A_x^2 \rangle \leqslant M_1 \qquad \forall x \in \mathbb{Z}^\nu \tag{4.13}$$

Proof. By Eq. (4.9)

$$\langle A_x^2 \rangle = \left\langle \int \lambda(dq'_x)\, \lambda(dq''_x)\, q'_x q''_x \exp\{-\beta[W(q'_x; q_{\mathbb{Z}^\nu/x}) + W(q''_x; q_{\mathbb{Z}^\nu/x}) - 2W(q_x; q_{\mathbb{Z}^\nu/x})]\} \right\rangle$$

Let us define

$$E_\Lambda = \{\mathbf{q} \,\|q_x|^2 \leqslant a \log_+ |x|, \quad \forall x \in \Lambda^c\}$$

where $\log_+ |x| = \max\{1, \log |x|\}$. The superstability estimate (2.1) easily implies[2] that, if a is sufficiently large, $\mu(E_\Lambda) \xrightarrow[\Lambda \nearrow \mathbb{Z}^\nu]{} 1$. Then

$$\langle A_x^2 \rangle \leqslant \lim_{\substack{\Lambda \nearrow \mathbb{Z}^\nu \\ \Lambda \ni x}} \int_{E_\Lambda} \mu[dq_{\Lambda^c}] \int \lambda(dq'_x)\, \lambda(dq''_x)\, |q'_x|\, |q''_x| \times Z_\Lambda(q_{\Lambda^c})^{-1} \exp[-\beta \tilde{H}(q_\Lambda, q'_x, q''_x \mid q_{\Lambda^c})] \tag{4.14}$$

where

$$\tilde{H}(q_\Lambda q'_x q''_x \mid q_{\Lambda^c}) = H(q_\Lambda \mid q_{\Lambda^c}) - 2W(q_x; q_{\mathbb{Z}^\nu/x}) + W(q'_x; q_{\mathbb{Z}^\nu/x}) + W(q''_x; q_{\mathbb{Z}^\nu/x})$$

$\tilde{H}$ can be thought as analogous to H for a new spin system obtained by adding two spins q'_x and q''_x. We denote by $\mathscr{L}_x$ the set $\mathbb{Z}^\nu \cup \{x', x''\}$, where x' and x'' are two copies of x, and by $\tilde{\mathbf{q}} = (\tilde{q}_z)_{z \in \mathscr{L}_x}$ a configuration of the new system. The interaction of this system is described by the two-body potential $\tilde{\phi}_{z_1 z_2}(q, q')$, where

$$\tilde{\phi}_{z_1 z_2}(q, q') = \begin{cases} \phi_{x z_2}(q, q') & \text{if } z_1 \in \{x', x''\}, \quad z_2 \in \mathbb{Z}^\nu / x \\ \phi_{z_1 z_2}(q, q') & \text{if } z_1, z_2 \in \mathbb{Z}^\nu / x \\ -\phi_{x z_2}(q, q') & \text{if } z_1 = x, \quad z_2 \in \mathbb{Z}^\nu / x \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that Eq. (4.6) implies for the new system the superstability condition (2.1).

Let us now consider the function

$$\rho_\Lambda(q_x, q'_x, q''_x) = Z_\Lambda(q_{\Lambda^c})^{-1} \int \lambda(dq_{\Lambda \setminus x}) \exp[-\beta \tilde{H}(q_\Lambda q'_x q''_x \mid q_{\Lambda^c})] \quad (4.15)$$

If $q_{\Lambda^c} \in E_\Lambda$, we can proceed as in the proof of Theorem 4.1 of Ref. 2 in order to show that, if Λ is sufficiently large,

$$\rho_\Lambda(q_x q'_x q''_x) \leqslant \exp[(\beta r - \gamma)(q_x^2 + q_x'^2 + q_x''^2) + 3\delta] \quad (4.16)$$

for a suitable δ, independent of q_{Λ^c} and Λ, and

$$\gamma < \beta[r - J/2 - J(1)]$$

The only difference with respect to Ref. 2 is that in (4.15) $Z_\Lambda(q_{\Lambda^c})$ is the normalization constant of the old spin system. However, it is possible to take this fact into account in a trivial way (we omit the details). (4.13) immediately follows from Eqs. (2.1), (4.14), and (4.16). ∎

Lemma 4.4. There exists a constant $M_2 > 0$ such that

$$|\langle A_\Lambda^2 \rangle - \langle Q_\Lambda^2 \rangle| \leqslant M_2 |\Lambda| \quad (4.17)$$

Proof. By some simple algebra

$$\begin{aligned} \langle A_\Lambda^2 \rangle - \langle Q_\Lambda^2 \rangle = & \sum_{x \in \Lambda} [\langle A_x^2 \rangle - \langle q_x^2 \rangle] \\ & + \sum_{\substack{x, y \in \Lambda \\ x \neq y}} \Big\langle \int \lambda(d\bar{q}_x)\, \lambda(d\bar{q}_y)\, q_x q_y \{\exp[\beta\phi(\bar{q}_x \bar{q}_y) \\ & + \beta\phi(q_x q_y) - \beta\phi(\bar{q}_x q_y) - \beta\phi(q_x \bar{q}_y)] - 1\} \Big\rangle \end{aligned}$$

Equations (2.1) and (2.13) imply that $\langle q_x^2\rangle \leqslant C_0$, $\forall x \in \mathbb{Z}^\nu$, for a suitable C_0. Then, using Lemma 4.3 and again Eq. (2.13),

$$|\langle A_\Lambda^2\rangle - \langle Q_\Lambda^2\rangle| \leqslant (C_0 + M_1)|\Lambda| + \sum_{x\in\Lambda}\sum_{y\neq x} C_{xy} \tag{4.18}$$

where

$$\begin{aligned} C_{xy} = e^{2\delta} \int & \lambda(d\bar{q}_x)\,\lambda(d\bar{q}_y)\,\lambda(dq_x)\,\lambda(dq_y)\,|q_x|\,|q_y| \\ & \times \exp[-(\gamma - r\beta)(q_x^2 + q_y^2)] \\ & \times \{\exp[\beta J(|x-y|)(q_x^2 + q_y^2 + \bar{q}_x^2 + \bar{q}_y^2)] - 1\} \\ \leqslant \beta e^{2\delta} J(|x-y|) & \int \lambda(d\bar{q}_x)\,\lambda(d\bar{q}_y)\,\lambda(dq_x)\,\lambda(dq_y)\,|q_x|\,|q_y| \\ & \times [q_x^2 + q_y^2 + \bar{q}_x^2 + \bar{q}_y^2]\exp\{-[\gamma - r\beta - \beta J(1)](q_x^2 + q_y^2) \\ & + \beta J(1)(\bar{q}_x^2 + \bar{q}_y^2)\} \end{aligned}$$

Equation (4.17) follows from (4.18), (4.6), and (2.1), if γ is near enough to $\beta(r - J/2)$ [see Eq. (2.1)].

Lemma 4.5. There exists a constant $M_3 > 0$ such that

$$\langle B_\Lambda\rangle \geqslant M_3|\Lambda| \tag{4.19}$$

Proof. Let x be an arbitrary point in $\mathbb{Z}^\nu$. By Eq. (2.4)

$$\mu(dq_x) = \lambda(dq_x)\int \mu[dq_{\mathbb{Z}^\nu/x}]\, Z_{\{x\}}(q_{\mathbb{Z}^\nu/x})^{-1}\exp[-\beta W(q_x; q_{\mathbb{Z}^\nu/x})]$$

Let

$$E_{B,a} = \{q_{\mathbb{Z}^\nu/x}\,|\,q_y^2 < B + a\log|x-y|\ \forall y\neq x\}$$

If $q_{\mathbb{Z}^\nu/x} \in E_{B,a}$, using Eqs. (2.3) and (2.4) we find

$$\begin{aligned} |W(q_x; q_{\mathbb{Z}^\nu/x})| &\leqslant \tfrac{1}{2}\sum_{y\neq x} J(|x-y|)[q_x^2 + B + a\log|y-x|] \\ &\leqslant \tfrac{1}{2}Jq_x^2 + C \end{aligned}$$

for a suitable $C > 0$. Then

$$\mu(dq_x) \geqslant \lambda(dq_x)\,e^{-\frac{1}{2}Jq_x^2 - C}\left[\int \lambda(dq)\,e^{\frac{1}{2}Jq^2 + C}\right]^{-1}\mu(E_{B,a})$$

By Eqs. (2.1) and (4.6), there exists a constant $D > 0$ such that

$$\mu(dq_x) \geqslant D\lambda(dq_x)\, e^{-\frac{1}{2}Jq_x^2}\, \mu(E_{B,a})$$

Furthermore, by Eq. (2.13)

$$\begin{aligned}\mu(E_{B,a}) &\geqslant 1 - \sum_{y \neq x} \int_{q_y^2 > B + a \log|x-y|} \lambda(dq_y)\, e^{\beta r q_y^2}\, e^{-\gamma q_y^2 + \delta} \\ &\geqslant 1 - \sum_{y \neq x} \int \lambda(dq_y)\, e^{\beta r q_y^2}\, e^{-\gamma q_y^2 + \delta}\, e^{\gamma(q_y^2 - B - a \log|x-y|)} \\ &\geqslant 1 - e^{\delta - \gamma B} \left[\int \lambda(dq)\, e^{\beta r q^2} \right] \sum_{y \neq x} \frac{1}{|x-y|^{\gamma a}}\end{aligned}$$

Then, if B and a are sufficiently large, there exists $E > 0$ independent of x, such that

$$\mu(dq_x) \geqslant E\lambda(dq_x)\, e^{-\frac{1}{2}Jq_x^2}$$

This immediately implies Eq. (4.19), if one takes in account Eq. (2.2). ∎

Proof of Theorem 4.1. By the Schwartz inequality

$$(\langle Q_\Lambda^2 - \langle Q_\Lambda \rangle^2)(\langle A_\Lambda^2 \rangle - \langle A_\Lambda \rangle^2) \geqslant (\langle Q_\Lambda A_\Lambda \rangle - \langle Q_\Lambda \rangle \langle A_\Lambda \rangle)^2 \tag{4.20}$$

Using (4.10) and (4.11), we can write

$$\langle A_\Lambda \rangle^2 = \langle Q_\Lambda \rangle^2 = \langle Q_\Lambda^2 \rangle - \Delta_\Lambda^2 = \langle Q_\Lambda A_\Lambda \rangle + \langle B_\Lambda \rangle - \Delta_\Lambda^2 \tag{4.21}$$

where

$$\Delta_\Lambda^2 = \langle Q_\Lambda^2 \rangle - \langle Q_\Lambda \rangle^2 \tag{4.22}$$

Inserting (4.21) and (4.22) in (4.20), we obtain

$$\Delta_\Lambda^2(\langle A_\Lambda^2 \rangle + \Delta_\Lambda^2 - \langle Q_\Lambda A_\Lambda \rangle - \langle B_\Lambda \rangle) \geqslant (\Delta_\Lambda^2 - \langle B_\Lambda \rangle)^2$$

i.e.,

$$\Delta_\Lambda^2(\langle A_\Lambda^2 \rangle - \langle Q_\Lambda A_\Lambda \rangle + \langle B_\Lambda \rangle) \geqslant \langle B_\Lambda \rangle^2$$

and, using again (4.11),

$$\Delta_\Lambda^2(\langle A_\Lambda^2 \rangle - \langle Q_\Lambda^2 \rangle + 2\langle B_\Lambda \rangle) \geqslant \langle B_\Lambda \rangle^2 \tag{4.23}$$

Equations (4.17) and (4.23) imply that

$$\Delta_\Lambda^2 \geqslant \frac{\langle B_\Lambda\rangle^2}{2\langle B_\Lambda\rangle + M_2 |\Lambda|}$$

which implies, by Eq. (4.19),

$$\chi_\Lambda = \frac{1}{|\Lambda|}\Delta_\Lambda^2 \geqslant \frac{(\langle B_\Lambda\rangle |\Lambda|^{-1})^2}{2\langle B_\Lambda\rangle |\Lambda|^{-1} + M_2} \geqslant \frac{M_3^2}{2M_3 + M_2} > 0 \quad \blacksquare$$

Theorem 4.2. Let $d(q)$ be a real function of the spin variable which is not constant on the support of $\lambda(dq)$ and such that $|d(q)| < C|q|$. Under the conditions of Theorem 3.1 there exists a constant $M > 0$ independent of Λ such that

$$|\Lambda|^{-1}[\langle D_\Lambda^2\rangle - \langle D_\Lambda\rangle^2] \geqslant M$$

where

$$D_\Lambda = \sum_{x \in \Lambda} d(q_x)$$

This theorem expresses the strict positivity of the generalized susceptibility

$$\chi_d = \lim_{\Lambda \to \mathbb{Z}^\nu} |\Lambda|^{-1}[\langle D_\Lambda^2\rangle - \langle D_\Lambda\rangle^2]$$

and is proved in the same manner as Theorem 4.1. One first introduces the inessential condition $\int \lambda(dq)\, d(q) = 0$ and then the quantities A_Λ and B_Λ replacing q by $d(q)$. Note that for translation invariant states

$$\chi_d = \sum_{y \in \mathbb{Z}^\nu} (\langle d(q_x)\, d(q_y)\rangle - \langle d(q_x)\rangle\langle d(q_y)\rangle) \tag{4.24}$$

if the sum is finite.

5. SUM RULES AND CLUSTERING PROPERTIES FOR INFINITE SYSTEMS

We consider an infinite system satisfying the conditions of Section 2 and we furthermore assume that the two-body potential is symmetric, translation invariant, and has a power-law decay at infinity with power γ, i.e., the condition (2.4) becomes

$$J(n) \leqslant \frac{k}{n^\gamma}, \qquad \gamma > \nu \tag{5.1}$$

and

$$\lim_{\lambda \to \infty} \lambda^{\gamma} \phi_{x+\lambda \hat{u}, x_1}(q, q_1) = d_{\hat{u}}(q, q_1) \tag{5.2}$$

with $\hat{u}$ some fixed unit vector and $d_{\hat{u}}(q, q_1)$ not identically zero.

It then follows from (2.3) that

$$|d_{\hat{u}}(q, q_1)| \leqslant c\,|q|\,|q_1| \tag{5.3}$$

If $\mu \in M_0(\Omega)$ and $X \subset \mathbb{Z}^{\nu}$ is a finite set, we define the correlation function of μ for the set X by

$$\mu(dq_X) = \lambda(dq_X)\, \rho_X(q_X) \tag{5.4}$$

In the following, to simplify the notation, we shall consider only translation invariant states. Let $d(q) = d_{\hat{u}}(q_0, q)$ with q_0 any fixed value for which $\rho_x(q_0) \neq 0$. In this section, we shall establish the following result.

Theorem 5.1. Assume that the two-body potential satisfies the conditions (5.1) and (5.2); then, for any (translation invariant) equilibrium state $\mu \in M_0(\Omega)$, the following sum rule

$$\sum_{y \in \mathbb{Z}^{\nu}} [\langle d(q_x)\, d(q_y)\rangle - \langle d(q_x)\rangle \langle d(q_y)\rangle] = 0$$

is satisfied whenever the clustering is faster than the decay of the potential, i.e., whenever for any finite sets X_1 and X_2

$$\lim_{\lambda \to \infty} \lambda^{\gamma} [\rho_{X_1, X_2 + \lambda \hat{u}} - \rho_{X_1} \rho_{X_2 + \lambda \hat{u}}] = 0 \tag{5.5}$$

together with

$$|\rho_{X_1, X_2}(q_{X_1}, q_{X_2}) - \rho_{X_1}(q_{X_1})\, \rho_{X_2}(q_{X_2})| \leqslant \frac{\xi^{|X_1| + |X_2|} \exp(A \sum_{x \in X_1 \cup X_2} q_x^2)}{d(X_1, X_2)^{\gamma} + 1} \tag{5.6}$$

where ξ and A are some constants, $A < \beta r$, and

$$d(X_1, X_2) = \min_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} |x_1 - x_2|$$

From Theorems 4.2 and 5.1, we immediately have the following theorem.

Theorem 5.2. Suppose the two-body potential has a power law decay satisfying (5.1) and (5.2) with a function $d_{\hat{u}}(q_1, q_2)$ which is not constant on the support of $\lambda(dq_1) \cdot \lambda(dq_2)$. Then for any translation invariant equilibrium state $\mu \in M_0(\Omega)$, the clustering (defined as in Theorem 5.1) cannot be faster than the decay of the potential.

To establish Theorem 5.1, we first remark that any equilibrium state $\mu \in M_0(\Omega)$ satisfies the following Kirkwood–Salzburg equations (see Appendix A):

$$\rho_{xX}(q_x q_X) = e^{-\beta W(q_x; q_X)} \sum_{Y \subset \mathbb{Z}^\nu / Xx} \int \lambda(dq_Y)\, K_{x;Y}(q_x; q_Y) \times \rho_{xXY}(q_x = 0, q_X, q_Y) \tag{5.7}$$

where X is any finite set, $x \notin X$, and

$$\begin{aligned} W(q_x; q_X) &= \sum_{y \in X} \phi_{xy}(q_x, q_y) \\ W(q_x; q_X) &= 0 \qquad \text{if} \quad X = \phi \\ K_{x;Y}(q_x; q_Y) &= \prod_{y \in Y} [e^{-\beta \phi_{xy}(q_x, q_y)} - 1] \\ K_{x;Y}(q_x; q_Y) &= 1 \qquad \text{if} \quad Y = \phi \end{aligned} \tag{5.8}$$

It then follows from (5.7) that

$$\begin{aligned} &\rho_{xx_1}(qq_1) - \rho_x(q)\, \rho_{x_1}(q_1) \\ &\quad = e^{-\beta \phi_{xx_1}(q, q_1)} \sum_{Y \subset \mathbb{Z}^\nu / xx_1} \int \lambda(dq_Y)\, K_{x;Y}(q; q_Y) \\ &\qquad \times [\rho_{xx_1Y} - \rho_{x_1} \rho_{xY}](q_x = 0, q_1, q_Y) \\ &\qquad + K_{x;x_1}(q; q_1) \sum_{Y \subset \mathbb{Z}^\nu / xx_1} \int \lambda(dq_Y)\, K_{x;Y}(q; q_Y)\, \rho_{x_1}(q_1)\, \rho_{xY}(q_x = 0, q_Y) \\ &\qquad - \rho_{x_1}(q_1) \int \lambda(d\bar{q}_1)\, K_{x;x_1}(q; \bar{q}_1)\, e^{\beta \phi_{xx_1}(q, \bar{q}_1)}\, \rho_{xx_1}(q, \bar{q}_1) \end{aligned} \tag{5.9}$$

In the following, we are going to study $\rho_{xx_1} - \rho_x \rho_{x_1}$ in the limit $|x - x_1| \to \infty$. Let

$$x^\lambda = x + \lambda \hat{u}$$

Using (5.2) and (5.9) we have

$$\lim_{\lambda\to\infty} \lambda^{\gamma}(\rho_{x^\lambda x_1} - \rho_{x^\lambda}\rho_{x_1})(q, q_1)$$

$$= \lim_{\lambda\to\infty} \sum_{Y\subset \mathbb{Z}^\nu/x^\lambda x_1} \lambda^{\gamma}\int \lambda(dq_Y)\, K_{x^\lambda;Y}(q; q_Y)$$

$$\times [\rho_{x^\lambda xY} - \rho_{x_1}\rho_{x^\lambda Y}](q=0, q_1 q_Y) \qquad \text{(I)}$$

$$-\beta d_{\hat{u}}(q, q_1) \lim_{\lambda\to\infty} \sum_{Y\subset \mathbb{Z}^\nu/x^\lambda x_1} \int \lambda(dq_Y)\, K_{x^\lambda;Y}(q; q_Y)$$

$$\times \rho_{x_1}(q_1)\, \rho_{x^\lambda Y}(q=0, q_Y) \qquad \text{(II)}$$

$$-\rho_{x_1}(q_1) \lim_{\lambda\to\infty} \lambda^{\gamma}\int \lambda(d\bar{q}_1)\, K_{x^\lambda;x_1}(q; \bar{q}_1)\, e^{\beta\phi_{x^\lambda x_1}(q,\bar{q}_1)}\, \rho_{x^\lambda x_1}(q\bar{q}_1) \qquad \text{(III)}$$

Therefore assuming the clustering condition (5.5) we have

$$0 = \text{(I)} + \text{(II)} + \text{(III)}$$

The idea to evaluate the right-hand side is to notice the following: In (I) only those Y_s with one point y near x_1 and the other points $Y/y = \tilde{Y}$ near x^λ will contribute; since

$$\lambda^{\gamma}K_{x^\lambda;y}(q; q_y) \to -\beta d_{\hat{u}}(q, q_y) \quad \text{and} \quad \rho_{x^\lambda x_1 Y} - \rho_{x_1}\rho_{x^\lambda Y} \to \rho^T_{x_1 y}\, \rho_{x^\lambda \tilde{Y}}$$

we obtain:

$$\text{(I)} = \sum_{\substack{y\in \mathbb{Z}^\nu/x_1 \\ \tilde{Y}\subset \mathbb{Z}^\nu/x}} \int \lambda(dq_y)[-\beta d_{\hat{u}}(q, q_y)] \int \lambda(dq_{\tilde{Y}})\, K_{x;\tilde{Y}}(q; q_{\tilde{Y}})$$

$$\times \rho^T_{x_1 y}(q_1 q_y)\, \rho_{x\tilde{Y}}(q=0, q_{\tilde{Y}})$$

$$= \sum_{y\in \mathbb{Z}^\nu/x_1} \int \lambda(dq_y)[-\beta d_{\hat{u}}(q, q_y)]\, \rho^T_{x_1 y}(q_1, q_y)\, \rho_x(q)$$

In (II) only those Y's near x^λ will contribute, i.e.,

$$\text{(II)} = -\beta d_{\hat{u}}(q, q_1)\, \rho_{x_1}(q_1)\, \rho_x(q)$$

Finally, using the clustering property

$$\text{(III)} = \beta\rho_{x_1}(q_1)\, \rho_x(q) \int \lambda(d\bar{q}_1)\, d_{\hat{u}}(q, \bar{q}_1)\, \rho_{x_1}(\bar{q}_1)$$

Therefore, assuming $\rho_x(q_0) \neq 0$ we obtain

$$0 = -\sum_{y \in \mathbb{Z}^\nu/x_1} \int \lambda(dq_y)\, d_{\hat{u}}(q_0, q_y)\, \rho^T_{x_1 y}(q_1, q_y) + \rho_{x_1}(q_1) \left[\int \lambda(d\bar{q}_1)\, d_{\hat{u}}(q_0, \bar{q}_1)\, \rho_{x_1}(\bar{q}_1) - d_{\hat{u}}(q_0, q_1) \right] \tag{5.10}$$

The rigorous deviation of (5.10) is given in Appendix B. Let then $d(q) = d_{\hat{u}}(q_0; q)$; multiplying (5.10) by $\int \lambda(dq_1)\, d(q_1)$, we obtain

$$\begin{aligned} 0 &= -\sum_{y \in \mathbb{Z}^\nu/x_1} \int \lambda(dq_1) \int \lambda(dq_y)\, d(q_1)\, d(q_y)\, \rho^T_{x_1 y}(q_1, q_y) \\ &\quad + \int \lambda(dq_1)\, d(q_1)\, \rho_{x_1}(q_1) \left[\int \lambda(d\bar{q}_1)\, d(\bar{q}_1)\, \rho_{x_1}(\bar{q}_1) - d(q_1) \right] \\ &= -\sum_{y \in \mathbb{Z}^\nu} [\langle d(q_{x_1})\, d(q_y) \rangle - \langle d(q_{x_1}) \rangle \langle d(q_y) \rangle] \end{aligned} \tag{5.11}$$

which concludes the proof.

Remark. Let us note that the condition on $d_{\hat{u}}(q_1, q_2)$ introduced in Theorem 5.2 is not very restrictive, since otherwise the asymptotic behavior of the potential would be independent of the spin variables.

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APPENDIX A

In this appendix, we shall derive the Kirkwood–Salzburg equations for the family $M_0(\Omega)$ of equilibrium states defined in Section 2. We need the following lemma.

Lemma A.1. If X is a finite set, $z \in X$, and $f(q_X)$ is any bounded measurable cylindrical function with base in X, then

$$\langle f \rangle = \lim_{\substack{\Lambda \nearrow \mathbb{Z}^\nu \\ z \in \Lambda}} \left\langle f \exp\left[\beta \sum_{y \in \Lambda^c} \phi_{z,y}(q_z, q_y)\right] \right\rangle \tag{A.1}$$

Proof. We will proceed as in the proof of Lemma 4.3. If $E_\Lambda = \{\mathbf{q} \mid |q_x|^2 \leqslant a \log_+ |x|, \quad \forall x \in \Lambda^c\}$ and a is sufficiently large, $\mu(\lim_{\Lambda \nearrow \mathbb{Z}^\nu} E_\Lambda) = 1$,[2] which implies that, for any fixed $q_z \in \mathbb{R}^d$

$$\sum_{y \in \Lambda^c} \phi_{z,y}(q_z, q_y) \xrightarrow[\substack{\Lambda \nearrow \mathbb{Z}^\nu \\ z \in \Lambda}]{} 0 \qquad (q_{\mathbb{Z}^\nu/X} \text{ a.e.})$$

By dominated convergence, it is then sufficient to show that

$$\left\langle |f| \exp\left[\sum_{y \neq z} \phi_{z,y}^+(q_z, q_y)\right]\right\rangle < \infty$$

where $\phi_{z,y}^+$ is the positive part of $\phi_{a,y}$. By Eq. (2.7)

$$\left\langle |f| \exp\left[\sum_{y \neq z} \phi_{z,y}^+(q_z, q_y)\right]\right\rangle$$

$$= \lim_{\substack{\Lambda \nearrow \mathbb{Z}^\nu \\ \Lambda \supset X}} \int \mu(dq_{\Lambda^c}) \int \lambda(dq_\Lambda) \, |f(q_X)| \, Z_\Lambda(q_{\Lambda^c})^{-1} \exp[-\beta \tilde{H}(q_\Lambda(q_{\Lambda^c})] \tag{A.2}$$

where $\tilde{H}$ is the analog of H for a spin system with the potential

$$\tilde{\phi}_{xy}(q, q') = \begin{cases} \phi_{xy}(q, q') & \text{if } \; x, y \neq z \\ \phi_{zy}(q, q') & \text{if } \; x = z \; \text{ and } \; \phi_{zy}(q, q') < 0 \\ 0 & \text{if } \; x = z \; \text{ and } \; \phi_{zy}(q, q') \geqslant 0 \end{cases}$$

Of course, this new spin system satisfies the superstability condition (2.1) with the same constants A and B of the old one. Then, by the same argument used in the proof of Lemma 4.3, if Λ is sufficiently large the integral in Eq. (A.2) can be bounded by a constant independent of Λ.

By Eqs. (A.1), (2.7), and (5.4), if $z \in X$

$$\rho_X(q_X) = \lim_{\substack{\Lambda \nearrow \mathbb{Z}^\nu \\ \Lambda \supset X}} \int \mu(dq_{\Lambda^c}) \int \lambda(dq_{\Lambda/X}) \, Z_\Lambda(q_{\Lambda^c})^{-1} \, e^{-\beta H(q_{\Lambda/z} | q_{\Lambda^c})}$$

$$\times \exp\left[-\beta \sum_{y \in X/z} \phi_{zy}(q_z, q_y)\right] \exp\left[-\beta \sum_{y \in \Lambda/X} \phi_{zy}(q_z, q_y)\right]$$

But $\phi_{xy}(0, q_y) = 0$, then $H(q_{\Lambda/z} \,|\, q_{\Lambda^c}) = H(q_{\Lambda/z} \, q_z = 0 \mid q_{\Lambda^c})$. Therefore, using the definitions (5.8)

$$\rho_X(q_X) = e^{-\beta W(q_z; q_{X/z})} \lim_{\substack{\Lambda \to \mathbb{Z}^\nu \\ \Lambda \supset X}} \sum_{Y \subset \Lambda/X} \int \lambda(dq_Y) \, K_{z;Y}(q_z; q_Y)$$

$$\times \int \mu(dq_{\Lambda^c}) \int \lambda(dq_{\Lambda/XY}) \, Z_\Lambda(q_{\Lambda^c})^{-1} \, e^{-\beta H(q_{\Lambda/z} q_z = 0 | q_{\Lambda^c} |)}$$

which implies that

$$\rho_{zX}(q_z q_X) = e^{-\beta W(q_z;q_X)} \sum_{Y \subset \mathbb{Z}^\nu/Xz} \int \lambda(dq_Y)\, K_{z;Y}(q_z; q_Y)\, \rho_{zXY}(q_z = 0, q_X, q_Y), \qquad z \notin X \tag{A.3}$$

The series in Eq. (A.3) is absolutely convergent. In fact, by Eq. (2.13)

$$\begin{aligned}
&\sum_{Y \subset \mathbb{Z}^\nu/zX} \int \lambda(dq_Y)\, |K_{z;Y}(q_z; q_Y)|\, \rho_{zXY}(q_z = 0, q_X, q_Y) \\
&\quad \leqslant \exp\left\{\delta(|X| + 1) + (\beta r - \gamma) \sum_{x \in X} q_x^2\right\} \\
&\quad \times \prod_{y \in \mathbb{Z}^\nu/Xz} \left\{1 + e^\delta \int \lambda(dq_y)\, |e^{-\beta\phi_{zy}(q_z, q_y)} - 1|\ e^{(\beta r - \gamma)q_y^2}\right\} \\
&\quad \leqslant \exp\left\{\delta(|X| + 1) + (\beta r - \gamma) \sum_{x \in X} q_x^2 + \sum_{y \neq z} e^\delta \beta J(|z - y|)\, |q_z|\right. \\
&\quad \times \left.\int \lambda(dq_y)\, e^{\beta r q_y^2}\, |q_y|\, e^{-\gamma q_y^2}\, e^{\beta J(1)\, |q_z|\, |q_y|}\right\}
\end{aligned} \tag{A.4}$$

which is finite, thanks to Eqs. (2.1), (2.4), and (4.6), if γ is suitably chosen. ∎

APPENDIX B

Contribution (II)

It is sufficient to show that (II) does not change if we make the substitution

$$\lim_{\Lambda \to \infty} \sum_{Y \subset \mathbb{Z}^\nu/x^\Lambda x_1} = \lim_{\Lambda \to \infty} \sum_{Y \subset \mathbb{Z}^\nu/x^\Lambda}$$

In fact, using the invariance under translation,

$$\lim_{\Lambda \to \infty} \sum_{Y \subset \mathbb{Z}^\nu/x^\Lambda} \int \lambda(dq_Y)\, K_{x^\Lambda; Y}(q; q_Y)\, \rho_{x^\Lambda Y}(q = 0, q_Y) = \rho_x(q)$$

Let us then consider the contribution due to those Y containing x_1:

$$\begin{aligned}
&\lim_{\Lambda \to \infty} \sum_{Y \subset \mathbb{Z}^\nu/x^\Lambda x_1} \int \lambda(dq_{x_1})\, K_{x^\Lambda; x_1}(q; q_{x_1}) \int \lambda(dq_Y)\, K_{x^\Lambda; Y}(q; q_Y) \\
&\quad \times \rho_{x^\Lambda x_1 Y}(q = 0, q_{x_1}, q_Y) \\
&= \lim_{\Lambda \to \infty} \int \lambda(dq_{x_1})\, K_{x^\Lambda; x_1}(q; q_{x_1})\, \rho_{x^\Lambda x_1}(q q_{x_1})
\end{aligned}$$

But, by Eq. (2.13)

$$|\rho_{x^\lambda x_1}(q, q_{x_1})| \leqslant e^{(\beta r-\gamma)(q^2+q_{x1}^2)} e^{2\delta}$$

and

$$|K_{x^\lambda;x_1}(q; q_{x_1})| \leqslant \beta J(|x^\lambda - x_1|)\,|q|\,|q_{x_1}|\, e^{\beta J(|x^\lambda - x_1|)\ |q|\ |q_{x_1}|}$$

which imply that the integrand is uniformly bounded by an integrable function; since the integrand converges pointwise to zero as $\lambda \to \infty$, we have concluded the proof of (II).

Contribution (III)

$$(\mathrm{III}) = -\rho_{x_1}(q_1) \lim_{\lambda \to \infty} \int \lambda(d\bar{q}_1) \lambda^\gamma K_{x^\lambda;x_1}(q;\bar{q}_1) e^{\beta\phi_{x^\lambda x_1}}(q;\bar{q}_1) \rho_{x^\lambda x_1}(q\bar{q}_1)$$

Since

$$|K_{x^\lambda;x_1}(q; \bar{q}_1)\, e^{\beta\phi_{x^\lambda x_1}(q,\bar{q}_1)}| \leqslant \beta J(|x^\lambda - x_1|)\,|q|\,|q_1|\, e^{\beta J(|x^\lambda - x_1|)\,|q|\,|\bar{q}_1|}$$

using (5.1) we can again apply the dominative convergence theorem to permute limit and integral.

Contribution (I)

(A) $Y = \phi$, $\quad \lambda^\gamma[\rho_{x^\lambda x_1} - \rho_{x^\lambda}\rho_{x_1}] \to 0$ by assumption

(B) $Y \neq \phi$

(B1) If $|y| > \lambda/8$ for all $y \in Y$, then there exists λ_0 such that, for $\lambda > \lambda_0$ $|y - x_1| > \lambda/16$, $|x^\lambda - x_1| > \lambda/16$. The contribution of these sets can be bounded by

$$\sum_{Y \subset \mathbb{Z}^\nu/x} \int \lambda(dq_Y)\,|K_{x;Y}(q; q_Y)|\,\lambda^\nu\,|\rho_{x^\lambda x_1, Y^\lambda} - \rho_{x_1}\rho_{x^\lambda Y^\lambda}|$$

$$\leqslant \sum_{Y \subset \mathbb{Z}^\nu/x} \int \lambda(dq_Y)\,|K_{x;Y}(q; q_Y)|\lambda^\gamma \frac{\xi^{|Y|+2}}{(\lambda/16)^\gamma} \exp\left[A\left(\sum_{y \in Y} q_Y^2 + q^2 + q_1^2\right)\right]$$

Proceeding as in the last part of Appendix A [see Eq. (A.4)] the integrand is thus bounded by an integrable function; applying dominated convergence this contribution yields zero.

(B2) The contribution of the sets $Y = \{y_1 \cdots y_n\}$ such that $|y_i| < \lambda/16$ and $|y_j| > \lambda/8$ for all $j \neq i$ can be bounded by

$$\sum_{\substack{|y_i| \leqslant \lambda/16 \\ y_i \neq x_1}} \sum_{\substack{\tilde{Y} \subset \mathbb{Z}^\nu/x^\lambda \\ |y_j| > \lambda/8}} \int \lambda(dq_i) \lambda^\gamma K_{x^\lambda; y_i}(q; q_i) \int \lambda(dq_Y) K_{x^\lambda; \tilde{Y}}(q; q_{\tilde{Y}}) \{\cdots\}$$

$$\{\cdots\} = \{[\rho_{x^\lambda x_1 y_i \tilde{Y}} - \rho_{x_1 y_i} \rho_{x^\lambda \tilde{Y}}] + \rho_{x^\lambda \tilde{Y}} [\rho_{x_1 y_i} - \rho_{x_1} \rho_{y_i}]$$
$$- \rho x_1 [\rho_{x^\lambda y_i \tilde{Y}} - \rho_{y_i} \rho_{x^\lambda \tilde{Y}}]\}$$

Once more, using Appendix A and (5.1), we can apply dominated convergence and only the second term in the bracket will yield a nonzero contribution.

(B3) If $\lambda/16 \langle |y_i| \leqslant \lambda/8$ and $|y_j| > \lambda/8$ for all $i \neq j$, then a similar argument shows that this contribution is zero since

$$\lambda^\gamma [\rho_{x^\lambda x_1 Y} - \rho_{x_1} \rho_{x^\lambda Y}] \to 0$$

(B4) All the contributions with more than one point y_i in Y such that $|y_i| < \lambda/8$ will give zero using similar arguments.

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